Kvantna mehanika2

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Kanonska transformacija in spinski valovi, A1

NAVODILO

Diagonaliziraj Hamiltonov operator za dva sklopljena harmonska oscilatorja:

$$H = \alpha A^\dagger A + \beta B^\dagger B + \gamma(A^\dagger B^\dagger + h.c.).$$

Napotek: predpostavi, da je H diagonalen v novi bazi. Iz enačb gibanja za operatorje v novi bazi sklepaj na obliko transformacije, ki diagonalizira H.

REŠITEV

Imamo dva oscilatorja, ki nimata nič skupnega oz. pripadata različnim Hilbertovim prostorom. Zato velja:

$$\{A, B\}_- = \{A^\dagger, B^\dagger\}_- = \{A, B^\dagger\}_- = \{A^\dagger, B\}_- = 0.$$

Velja pa seveda še:

$$\{A, A\}_- = \{A^\dagger, A^\dagger\}_- = \{B, B\}_- = \{B^\dagger, B^\dagger\}_- = 0$$

ter

$$\{A, A^\dagger\}_- = \{B, B^\dagger\}_- = 1.$$

Tako pa lahko zapišemo naš Hamiltonjan v obliki (upoštevamo gornje relacije in to, da h.c. pomeni hermitsko konjugiran člen $A^\dagger B^\dagger$):

$$H = \alpha A^\dagger A + \beta B^\dagger B + \gamma(A^\dagger B^\dagger + AB) \ ,$$

kjer so α, β, γ konstante.

Vemo, da je novi Hamiltonjan diagonalen. Torej ga lahko zapišemo v obliki:

$$H_N = a\mathcal{A}^\dagger \mathcal{A} + b\mathcal{B}^\dagger \mathcal{B} + c \ ,$$

kjer so a,b,c konstante.

Zahtevamo, da za operatorje $\mathcal{A}, \mathcal{A}^\dagger, \mathcal{B}, \mathcal{B}^\dagger$ veljajo enake komutacijske relacije kot za operatorje $A, A^\dagger, B, B^\dagger$ oz.

$$\{\mathcal{A}, \mathcal{A}^\dagger\}_- = \{\mathcal{B}, \mathcal{B}^\dagger\}_- = 1.$$

Vsi ostali komutatorji pa so enaki nič. Nove operatorje lahko zapišemo kot linearno kombinacijo starih:

$$\mathcal{A} = dA + eA^\dagger + fB + gB^\dagger$$
$$\mathcal{A}^\dagger = d^*A^\dagger + e^*A + f^*B^\dagger + g^*B$$
$$\mathcal{B} = hA + jA^\dagger + kB + lB^\dagger$$
$$\mathcal{B}^\dagger = h^*A^\dagger + j^*A + k^*B^\dagger + l^*B.$$

Komutatorji so torej enaki:

$$\{\mathcal{A}, \mathcal{A}^\dagger\}_- = |d|^2 - |e|^2 + |f|^2 - |g|^2 = 1$$
$$\{\mathcal{B}, \mathcal{B}^\dagger\}_- = |h|^2 - |j|^2 + |k|^2 - |l|^2 = 1.$$

Če želimo zapisati H_N v stari bazi, moramo le vstaviti $\mathcal{A}, \mathcal{A}^\dagger, \mathcal{B}, \mathcal{B}^\dagger$ izražene z $A, A^\dagger, B, B^\dagger$ v novi Hamiltonjan. Tako dobimo:

$$H_N = a[|d|^2 A^\dagger A + e^* d\boxed{\text{AA}} + f^* d\boxed{B^\dagger A} + g^* dBA +$$
$$+ d^* e\boxed{A^\dagger A^\dagger} + |e|^2 AA^\dagger + f^* eB^\dagger A^\dagger + g^* e\boxed{BA^{dag}} +$$
$$+ d^* f\boxed{A^\dagger B} + e^* fAB + |f|^2 B^\dagger B + g^* f\boxed{\text{BB}} +$$
$$+ d^* gA^\dagger B^\dagger + e^* g\boxed{AB^\dagger} + f^* g\boxed{B^\dagger B^\dagger} + |g|^2 BB^\dagger] +$$
$$+b[|h|^2 A^\dagger A + j^* h\boxed{\text{AA}} + k^* h\boxed{B^\dagger A} + l^* hBA +$$
$$+ h^* j\boxed{A^\dagger A^\dagger} + |j|^2 AA^\dagger + k^* jB^\dagger A^\dagger + l^* j\boxed{BA^{dag}} +$$
$$+ h^* k\boxed{A^\dagger B} + j^* kAB + |k|^2 B^\dagger B + l^* k\boxed{\text{BB}} +$$
$$+ h^* lA^\dagger B^\dagger + j^* l\boxed{AB^\dagger} + k^* l\boxed{B^\dagger B^\dagger} + |l|^2 BB^\dagger] +$$
$$+c.$$

Obkroženi členi ne nastopajo v starem hamiltonjanu H, s katerim primerjamo H_N razvit po stari bazi, tako da lahko postavimo določene koeficiente na nič. Rekla sem torej, da $e = e^* = 0$, $f = f^* = 0$, $h = h^* = 0$ ter $l = l^* = 0$. Torej so novi operatorji razviti po starih oblike:

$$\mathcal{A} = dA + gB^\dagger$$
$$\mathcal{A}^\dagger = d^* A^\dagger + g^* B$$
$$\mathcal{B} = jA^\dagger + kB$$
$$\mathcal{B}^\dagger = j^* A + k^* B^\dagger.$$

Transformacijo lahko tako zapišemo: $\begin{pmatrix} \mathcal{A} \\ \mathcal{A}^\dagger \\ \mathcal{B} \\ \mathcal{B}^\dagger \end{pmatrix} = \begin{pmatrix} d & 0 & 0 & g \\ 0 & d^* & g^* & 0 \\ 0 & j & k & 0 \\ j^* & 0 & 0 & k^* \end{pmatrix} \begin{pmatrix} A \\ A^\dagger \\ B \\ B^\dagger \end{pmatrix}$

oz. $\begin{pmatrix} \mathcal{A} \\ \mathcal{B}^\dagger \end{pmatrix} = \begin{pmatrix} d & g \\ j^* & k^* \end{pmatrix} \begin{pmatrix} A \\ B^\dagger \end{pmatrix} = S_2 \begin{pmatrix} A \\ B^\dagger \end{pmatrix}.$

Komutatorji sedaj dajo:

$$\{\mathcal{A}, \mathcal{A}^\dagger\}_- = |d|^2 - |g|^2 = 1 \ \Rightarrow \boxed{|d|^2 = 1 + |g|^2}$$
$$\{\mathcal{B}, \mathcal{B}^\dagger\}_- = |k|^2 - |j|^2 = 1 \ \Rightarrow \boxed{|k|^2 = 1 + |j|^2}\ .$$

Sam Hamiltonjan pa lahko zapišemo kot:

$$H_N = a[|d|^2 A^\dagger A + g^* dBA + d^* gA^\dagger B^\dagger + |g|^2 BB^\dagger] +$$
$$+ b[|j|^2 AA^\dagger + k^* jB^\dagger A^\dagger + j^* kAB + |k|^2 B^\dagger B] +$$
$$+ c =$$
$$= A^\dagger A[a|d|^2 + b|j|^2] + B^\dagger B[a|g|^2 + b|k|^2] +$$
$$+ A^\dagger B^\dagger [ad^* d + bk^* j] + AB[ag^* d + bj^* k] +$$
$$+ c[+a|g|^2 + b|j|^2.$$

Velja: $H = H_N(po\ stari\ bazi)$. Tako dobimo naslednje relacije (ko enačimo koeficiente pred istimi operatorji):

$$\alpha = a|d|^2 + b|j|^2$$
$$\beta = a|g|^2 + b|k|^2$$
$$\gamma = ad^* g + bk^* j = ag^* d + bj^* k$$
$$c + a|g|^2 + b|j|^2 = 0.$$

Iz tretje relacije dobimo, da so vsi koeficienti realni. To ter zveze dobljene iz komutatorjev $\{\mathcal{A}, \mathcal{A}^\dagger\}_-$ ter $\{\mathcal{B}, \mathcal{B}^\dagger\}_-$ upoštevamo v transformaciji S_2:

$$S_2 = \begin{pmatrix} \pm\sqrt{1 + |g|^2} & g \\ j & \pm\sqrt{1 + |j|^2} \end{pmatrix}.$$

Vemo, da velja $ch^2x - sh^2x = 1$ oz. uganemo, da je $g = j = shx$ oz. da se transformacija glasi:

$$S_2 = \begin{pmatrix} \pm chx & shx \\ shx & \pm chx \end{pmatrix}.$$

Determinanta te matrike je vedno enaka 1, torej je transformacija Lorentzova s čimer smo zadovoljni.

Torej lahko zapišemo:

$g = j = shx$
$d = k = chx$
$\alpha = j^2(a+b) + a = sh^2x(a+b) + a$
$\beta = j^2(a+b) + b = sh^2x(a+b) + b$
$\gamma = jk(a+b) = shxchx(a+b)$
$c = -j^2(a+b) = -sh^2x(a+b)$

Ko vse te relacije preoblikujemo, dobimo:

$\boxed{x = \frac{1}{2}arcth(\frac{\alpha+\beta}{2\gamma})}$ ter

$a = \alpha - \gamma th(\frac{1}{2}arcth(\frac{\alpha+\beta}{2\gamma}))$
$b = \beta - \gamma th(\frac{1}{2}arcth(\frac{\alpha+\beta}{2\gamma}))$
$c = -\gamma th(\frac{1}{2}arcth(\frac{\alpha+\beta}{2\gamma}))$ in

$g = j = sh(\frac{1}{2}arcth(\frac{\alpha+\beta}{2\gamma}))$
$d = k = ch(\frac{1}{2}arcth(\frac{\alpha+\beta}{2\gamma}))$